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Semianalytical Differential Quadrature Solution for Free Vibration Analysis of Rectangular Plates

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In this paper, the analysis of the titled problem is based on classical thin plate theory, and its numerical solution is carried out by a semianalytical differential quadrature method. The thin rectangular plates considered herein are simply supported on two opposite edges. The boundary conditions at the other two edges may be quite general, and between these two edges the plates may have varying thickness. However, the results contained in this paper are for plates that are elastically restrained against rotation at these edges and have linearly tapered thickness. On the basis of comparison with the available results in the published literature, it is believed that this solution method guarantees high numerical accuracy for the problem. Moreover, the computation times involved in the evaluation of free vibration characteristics are sufficiently small indicating that the solution method may possibly be further developed for the real-time analysis and design of vibrating plate systems.

Nomenclature

$A_{ij}^{(r)}$ = weighting coefficients of the nth-order derivative associated with the $X = X_i$ sampling point, $i, j = 1, 2, \ldots, N$
a, b = side lengths of the plate along the x and y axes
D = flexural stiffness component, used with subscript(s)
$\bar{D}$ = dimensionless flexural stiffness component, used with subscript(s)
$\bar{D}_t$ = dimensionless twisting stiffness, defined in Eq. (3)
D_0 = reference value of flexural stiffness, defined in Eq. (4)
E = elastic modulus, used with subscript(s)
F = function defining mode shape in the x direction, $F(X)$
F_i = $F(X_i)$
G = shear modulus, used with subscripts
H = dimensionless plate thickness, h/h_0
H', H'' = functions defined in Eq. (1a)
H_1 = value of H at $X = 1$
h = plate thickness, $h(x)$
h_0 = reference value of plate thickness
$[I]$ = identity matrix
K_θ = dimensionless rotational stiffness, $k_\theta a/D_0$
k_θ = rotational stiffness of the restraint at the plate edges
L_{ij} = linear differential operator defined in Eq. (14)
m, n = number of half-waves in the x and y directions, respectively
N = number of sampling points
W = dimensionless mode function of the vibrating plate in the lateral direction, $W(X, Y)$
X, Y = dimensionless coordinates, x/a and y/b
x, y = rectangular coordinates parallel to the plate edges
α = taper ratio
β = $n\pi\lambda$
λ = aspect ratio, a/b
ν = Poisson's ratio, used with subscripts
ρ = density of the plate material
Ω = dimensionless frequency, same as Ω_{mn}
ω = circular frequency of free vibrations, rad/s

Subscripts

L = longitudinal axis, along the major elastic modulus of orthotropic material
T = transverse axis, along the minor elastic modulus of orthotropic material
x, y = components of elastic constants and flexural stiffnesses

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Introduction

CURRENT research activities on the free vibration characteristics of plates and plate systems focus invariably on various complicating effects, such as thickness nonuniformity, shear deformation and rotary inertia effects, general polygonal boundaries, material anisotropy, and so on. The free vibration characteristics of thin rectangular plates of uniform thickness may be obtained by simply specifying the plate aspect ratio and, in addition, Poisson's ratio, if free edges are involved. The vibration characteristics of such plates have been studied at great lengths.[1] Generally speaking, however, with the inclusion of one or more complicating effects, the combinations of the geometric and/or material parameters increase enormously, and the task of generating a vibration characteristics database becomes quite a tedious and unmanageable task. For design and analysis purposes, the database may be utilized for the interpolation of characteristics corresponding to the parameters of the problem of interest. However, the characteristics so obtained may be lacking in accuracy, and the interpolation process itself may be time consuming. Moreover, the database itself costs computer storage. A better alternative to handle this situation is to have accurate and efficient numerical solution techniques which may be used for real-time analysis and design. The present work is actually motivated by such needs, and to this effect, a semianalytical differential quadrature solution for the free vibration analysis of rectangular plates is offered. The complicating effects considered herein are of variable thickness and of material variability in that the plates could be isotropic or specially orthotropic.

The differential quadrature method (DQM) was proposed in the early seventies by Bellman and Casti[2] and Bellman et al.[3] as an accurate and fast computing numerical solution technique for nonlinear partial differential equations of initial value problems. Later developments in the DQM were brought about by its application to physical problems. Thus, one may see, for example, the application of DQM in such diverse areas as nonlinear diffusion,[4] transport processes,[5] solution of Boltzmann equation,[6] structural mechanics,[7–9] solution of Navier–Stokes equations for flow past a cylinder[10] and a driven cavity problem,[11] convection heat transfer,[12] lubrication problems,[13] and so on. Some recent works[13,14] have focused on the assessment of the numerical accuracy and computational efficiency of the DQM. It has been shown that in both these respects DQM stands out in comparison to the conventional numerical solution techniques of the finite difference and finite element methods. In fact, the differential quadrature method is now recognized as a numerical technique for the solution of differential equations; besides three books of Bellman,[15] Bellman and Adomian,[16] and Bellman and Roth,[17] the method has been recently included in a handbook of differential equations.[18] The applications of the DQM in wide ranging engineering problems and high numerical accuracy and computational efficiency offer the DQM as an alternative technique to the conventional techniques of finite element and finite difference methods. Recently, utilizing the domain decomposition concept, applications of the DQM to larger class of structural mechanics problems have been offered,[19,20] which are indicative of the future possibility of the DQM becoming a competitor to the finite element method.

The rectangular plate configurations considered for the application of the present work are taken to be simply supported on two opposite edges. The boundary conditions at the other two edges may be quite general, and between these two edges, the plates may have varying thickness. Nevertheless, the results contained in this paper are for plates which are elastically restrained against rotation at these edges and have linearly tapered thickness.

In the following section, the governing equations of the problem and the details of the semianalytical differential quadrature solution are given. Next, the results pertaining to the objectives of the work are presented.

Governing Equations and Solution Procedure

Consider a thin rectangular plate of an orthotropic material having its sides and the axes of material symmetry aligned with the x and y axes. Let the thickness h of the plate be varying in the x direction only, that is, $h = h(x)$. The governing eigenvalue differential equation of the plate undergoing free harmonic vibrations may be written in a nondimensional form as

$$H^2\bar{D}_x\frac{\partial^4 W}{\partial X^4} + 2\lambda^2 H^2(\bar{D}_{xy} + 2\bar{D}_t)\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \lambda^4 H^2 \bar{D}_y \frac{\partial^4 W}{\partial Y^4}$$
$$+ 2H'\bar{D}_x\frac{\partial^3 W}{\partial X^3} + 2\lambda^2 H'(\bar{D}_{xy} + 2\bar{D}_t)\frac{\partial^3 W}{\partial X \partial Y^2}$$
$$+ H''\left(\bar{D}_x\frac{\partial^2 W}{\partial X^2} + \lambda^2\bar{D}_{xy}\frac{\partial^2 W}{\partial Y^2}\right) = \Omega^2 W \tag{1}$$

where

$$H' = \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}X}(H^3), \qquad H'' = \frac{1}{H}\frac{\mathrm{d}^2}{\mathrm{d}X^2}(H^3) \tag{1a}$$

In Eq. (1), the dimensionless frequency Ω is defined as

$$\Omega^2 = \left(\rho h_0 a^4/D_0\right)\omega^2 \tag{2}$$

The dimensionless flexural and twisting stiffnesses of the plate are defined as

$$\bar{D}_x, \bar{D}_y = \frac{(E_x, E_y)h_0^3}{12(1 - \nu_{xy}\nu_{yx})D_0}, \qquad \bar{D}_{xy} = \nu_{yx}\bar{D}_x = \nu_{xy}\bar{D}_y$$
$$\bar{D}_t = \frac{G_{xy}h_0^3}{12D_0} \tag{3}$$

The reference value of the flexural stiffness D_0 is taken as

$$D_0 = \left[E_L h_0^3/12(1 - \nu_{LT}\nu_{TL})\right] \tag{4}$$

where L and T directions may be coincident with either of the coordinate axes. In the results to follow, plate orientation having the longitudinal axis coincident with the x axis is referred to as the L/T orientation. In the T/L orientation, the transverse axis is coincident with the x axis. The expressions for the dimensionless stiffnesses for the two types of orientations, in terms of the orthotropic elastic constants are given in Table 1. Equation (1) applies to isotropic materials as well, with $\bar{D}_x = \bar{D}_y = 1$, $\bar{D}_{xy} = \nu$, and $\bar{D}_t = (1-\nu)/2$.

Table 1 Dimensionless stiffnesses for specially orthotropic plates

Orientation	$\bar{D}_x$	$\bar{D}_y$	$\bar{D}_{xy}$	$\bar{D}_t$
L/T	1	E_T/E_L	ν_{TL}	$(1-\nu_{LT}\nu_{TL})G_{LT}/E_L$
T/L	E_T/E_L	1	ν_{TL}	$(1-\nu_{LT}\nu_{TL})G_{LT}/E_L$

The particular plate configurations under consideration are the ones having the two y edges, $Y = 0$ and 1 simply supported. The boundary conditions at these edges are given by (at $Y = 0$ and $Y = 1$)

$$W = 0, \qquad \frac{\partial^2 W}{\partial Y^2} = 0 \tag{5}$$

The two x edges, $X = 0$ and 1, are assumed to have supports which provide elastic restraint against rotation. The boundary conditions at these edges are given by (at $X = 0$)

$$W = 0, \qquad \bar{D}_x\frac{\partial^2 W}{\partial X^2} - K_\theta\frac{\partial W}{\partial X} = 0 \tag{6}$$

and at $X = 1$

$$W = 0, \qquad \bar{D}_x\frac{\partial^2 W}{\partial X^2} + \frac{K_\theta}{H_1^3}\frac{\partial W}{\partial X} = 0 \tag{7}$$

Following the usual procedure of the differential quadrature solutions for the two-dimensional problems, one would have for the present problem a two-dimensional quadrature grid of $N \times N$ sampling points in the square domain $0 \le X \le 1$, $0 \le Y \le 1$. (This is by assuming the number of sampling points to be the same in the x and y directions; however, the number of points may be different in the two directions.) One would then write the quadrature analog of eigenvalue differential equation (1) at all the sampling points of the domain and implement the boundary conditions through the quadrature analog of Eqs. (5–7). The resulting algebraic equations so obtained would be of size $N^2 \times N^2$. This size may be substantially large, especially if higher mode frequencies are desired. In the present quadrature solution, a semianalytical approach is adopted wherein one spatial variable is eliminated, and the partial derivatives in the governing equations are reduced to ordinary derivatives. Consequently, for the quadrature solution, only a one-dimensional grid of N sampling points is required, and the algebraic equations obtained from the quadrature analog of the governing ordinary differential equations are reduced to a size of $N \times N$ only.

For the purpose of semianalytical solution, the mode function $W = W(X, Y)$ is written in the following form which satisfies the boundary conditions of the simple supports at $Y = 0$ and 1:

$$W = F(X)\sin n\pi Y \tag{8}$$

where n is an integer and represents the number of half-waves in the y direction of the vibrating plate and the function $F(X)$ defines the mode shape in the x direction. It is noted that the form of Eq. (8) is used for the analytical solution of free vibration of constant thickness rectangular plates having two opposite edges simply supported.[1]

Now, using Eq. (8), the eigenvalue partial differential equation (1) is reduced to the following ordinary differential equation:

$$H^2\bar{D}_x\frac{\mathrm{d}^4 F}{\mathrm{d}X^4} + 2H'\bar{D}_x\frac{\mathrm{d}^3 F}{\mathrm{d}X^3} + \left\{H''\bar{D}_x - 2\beta^2 H^2\right.$$
$$\left.\times(\bar{D}_{xy} + 2\bar{D}_t)\right\}\frac{\mathrm{d}^2 F}{\mathrm{d}X^2} - 2\beta^2 H'(\bar{D}_{xy} + 2\bar{D}_t)\frac{\mathrm{d}F}{\mathrm{d}X}$$
$$+ \left(\beta^4 H^2\bar{D}_y - \beta^2 H''\bar{D}_{xy}\right)F = \Omega_{mn}^2 F \tag{9}$$

where the dimensionless frequency is denoted by Ω_{mn} as the one associated with the mn mode, m being the number of half-waves in the x direction of the vibrating plate. It may seen that the solution domain of Eq. (9) is a line of unit length $0 \le X \le 1$.

Using Eq. (8) in Eqs. (6) and (7), the boundary conditions at the end points of the domain $0 \le X \le 1$ are also reduced at $X = 0$ to the following form having ordinary derivatives:

$$F = 0, \qquad \bar{D}_x\frac{\mathrm{d}^2 F}{\mathrm{d}X^2} - K_\theta\frac{\mathrm{d}F}{\mathrm{d}X} = 0 \tag{10}$$

and at $X = 1$ to

$$F = 0, \qquad \bar{D}_x \frac{d^2F}{dX^2} + \frac{K_\theta}{H_1^3}\frac{dF}{dX} = 0 \tag{11}$$

The eigenvalue problem is now described by Eqs. (9–11). For the differential quadrature solution of these equations, consider a set of N prespecified sampling points X_i and the corresponding function values $F_i = F(X_i); i = 1, 2, \ldots, N$ in the solution domain $0 \leq X \leq 1$. The value of an rth-order function derivative at a point X_i may be expressed by the quadrature rule as

$$\left.\frac{d^rF}{dX^r}\right|_{X=X_i} = \sum_{j=1}^{N} A_{ij}^{(r)} F_j \tag{12}$$

where $A_{ij}^{(r)}$ are the weighting coefficients of the rth-order derivative associated with the ith sampling point. These weighting coefficients may be obtained by having appropriate approximations (the test functions) of the function $F(X)$; the details of same may be found in the literature.[3]

Using the quadrature rule, Eq. (13), for various order derivatives in Eq. (9), one obtains the quadrature analog of the differential equation as the following set of linear algebraic equations:

$$\sum_{j=1}^{N} L_{ij}F_j + \left(\beta^4 H^2 \bar{D}_y - \beta^2 H'' \bar{D}_{xy}\right)F_i = \Omega_{mn}^2 F_i$$

$$i = 3, 4, \ldots, (N-2) \tag{13}$$

where

$$L_{ij} = H_i^2 \bar{D}_x A_{ij}^{(4)} + 2H_i' \bar{D}_x A_{ij}^{(3)} + \left\{H_i'' \bar{D}_x - 2\beta^2 H_i^2 \times (\bar{D}_{xy} + 2\bar{D}_t)\right\} A_{ij}^{(2)} - 2\beta^2 H_i'(\bar{D}_{xy} + 2\bar{D}_t)A_{ij}^{(1)} \tag{14}$$

It may be noted that the quadrature analog of the differential equation (13) is utilized to obtain $(N-4)$ equations only, since two points on both sides, the end point and its immediate adjacent point, have been omitted. The remaining four equations are actually obtained from the quadrature analog equations of the two boundary conditions at each end and, thereby, boundary conditions are invoked. The quadrature analog of the boundary conditions, Eqs. (10) and (11), may be obtained as

$$F_i = 0 \quad \text{and} \quad \bar{D}_x \sum_{j=1}^{N} \left\{A_{ij}^{(2)} - K_\theta A_{ij}^{(1)}\right\} F_j = 0, \qquad i = 1 \tag{15}$$

and

$$F_i = 0 \quad \text{and} \quad \bar{D}_x \sum_{j=1}^{N} \left\{A_{ij}^{(2)} + \frac{K_\theta}{H_1^3} A_{ij}^{(1)}\right\} F_j = 0, \qquad i = N \tag{16}$$

Note that in each set of these quadrature analog equations, weighting coefficients correspond to the boundary point itself. Each pair of these equations simply complements the quadrature analog equations of the differential equation at the boundary point and at the point adjacent to that boundary point.

The set of quadrature analog equations from the differential equation and the boundary conditions yield a system of N algebraic equations which may be written in matrix form as

$$\begin{bmatrix} [S_{bb}] & [S_{bd}] \\ [S_{db}] & [S_{dd}] \end{bmatrix} \begin{Bmatrix} \{F_b\} \\ \{F_d\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \Omega_{mn}^2 \{F_d\} \end{Bmatrix} \tag{17}$$

in which $\{F_d\}$ is a $(N-4) \times 1$ column vector of the function values F_i at the sampling points $i = 3, 4, \ldots, (N-2)$. The column vector $\{F_b\}$ is of size 4×1 comprising the four function values F_i at the sampling points $i = 1, 2, (N-1)$, and N. By eliminating the $\{F_b\}$ column from Eq. (14), one obtains the following eigenvalue equation:

$$[S]\{F_d\} - \Omega_{mn}^2 [I]\{F_d\} = 0 \tag{18}$$

in which the matrix $[S]$ is of the size $(N-4) \times (N-4)$. The eigenvalues, the frequency squared values, of the $[S]$ matrix are obtained by inverse iteration with shifting.[21] The solution yields, along with the eigenvalues, the corresponding eigenvector $\{F_d\}$ from which the number of half-waves m in the x direction become known. As is obvious, one needs to specify the half-waves n in the y direction as input data for the construction of the quadrature analog equations. Thus, at the end of the solution, one gets to know the free vibration characteristics in terms of the frequency Ω_{mn} and the corresponding mode pattern mn of the vibrating plate.

The convergence and accuracy of the quadrature solutions depend largely on the accuracy of the weighting coefficients and the spacing of sampling points; detailed investigations on these issues may be found in the differential quadrature literature.[22–24] The weighting coefficients are obtained most accurately by the use of explicit formulas.[10,22] A natural choice is to use equally spaced grid points. However, very often unequally spaced sampling points, such as those positioned at the zeros of orthogonal polynomials, offer a better choice with respect to both the accuracy and convergence characteristics of the quadrature solutions.[22–24] One particular choice, which the present investigators have experienced to be generally efficacious in a number of vibration analysis problems,[25,26] is of the so-called Chebyshev–Gauss–Lobatto points; the same have been used in the present work as well. These are given by

$$x_i = \frac{1 - \cos[(i-1)\pi/(N-1)]}{2}; \qquad i = 1, 2, \ldots, N \tag{19}$$

As explained earlier, the eigenvector in Eq. (18) does not contain the function values at the points $i = 1, 2, (N-1)$, and N. Inasmuch as the convergence and accuracy of iteration scheme for the eigenvalues depend on how closely the eigenvectors represent the actual mode shapes, the elimination of these four function values is an important matter. It is desirable that the boundary and its adjacent points be as close as possible so that after elimination, in effect, only two function values are absent from the eigenvectors. Thus, if the adjacent point at each boundary is at a small distance $\delta \sim 10^{-5}$, then the sampling points of Eq. (19) are modified as

$$\xi_1 = 0, \qquad \xi_2 = \delta, \qquad \xi_{N-1} = 1 - \delta, \qquad \xi_N = 1$$

$$\xi_i = \frac{1 - \cos[(i-1)\pi/(N-3)]}{2}; \qquad i = 3, 4, \ldots, (N-2) \tag{20}$$

In fact, these are the types of sampling points used in the present work.

Results and Discussion

The semianalytical quadrature solution approach presented in the foregoing section is employed for the evaluation of free vibration characteristics of linearly tapering plates. The dimensionless thickness function is taken in the form

$$H = 1 + \alpha X \tag{21}$$

where α is actually the taper ratio $(h_1 - h_0)/h_0$, h_0 and h_1 being the thicknesses of the plate at the ends $X = 0$ and $X = 1$, respectively. For a uniform thickness plate, $\alpha = 0$. The dimensionless stiffnesses needed for the analysis of specially orthotropic plates were based on the following mechanical properties of E-glass/epoxy material[27]:

$$E_L = 60.7 \text{ GPa}, \qquad E_T = 24.8 \text{ GPa}, \qquad G_{LT} = 12.0 \text{ GPa}$$

$$\nu_{LT} = 0.23$$

Two basic issues for adopting a numerical solution scheme are its convergence and the accuracy of its results. These issues were addressed comprehensively in another work[25] in which the free vibration problems of isotropic and orthotropic rectangular plates with all possible combinations of simply supported, clamped, and free boundary conditions at the x edges were considered. Of course, the boundary conditions at the y edges were simply supported. The solution method was found to exhibit monotonic convergence with

Table 2 Convergence of semianalytical DQ solution, α = 0.4 and K_θ = 10

			Number of sampling points, N				
			7	11	15	19	23
λ	m	n	Dimensionless frequency Ω_{mn} Isotropic plates, $\nu = 0.3$				
2/3	1	1	22.323276	22.427926	22.427917	22.427917	22.427917
	3	1	120.03612	121.31906	121.39382	121.39380	121.39380
3/2	1	1	41.121997	41.234036	41.234021	41.234021	41.234021
	2	2	155.59147	157.07696	157.08229	157.08231	157.08232
			Specially orthotropic plates (L/T orientation)				
2/3	1	1	20.369360	20.490974	20.490967	20.490967	20.490967
	3	1	117.27166	118.83273	118.91438	118.91436	118.91436
3/2	1	1	30.781950	30.858766	30.858752	30.858752	30.858752
	3	2	174.60491	170.88764	170.83709	170.83711	170.83711
			Specially orthotropic plates (T/L orientation)				
2/3	1	1	16.778053	16.886568	16.886560	16.886560	16.886560
	3	2	97.052329	94.622015	94.662609	94.662596	94.662596
3/2	1	1	34.821415	34.936896	34.936866	34.936866	34.936866
	2	2	130.52594	134.09137	134.09251	134.09253	134.09253

Table 3 Accuracy of semianalytical DQ solution; isotropic plates, ν = 0.3

	Dimensionless frequency Ω^*_{mn} and mode numbers m n					
λ	$\alpha = 0.4$, $K^*_\theta = 1$			$\alpha = 1.0$, $K^*_\theta = 100$		
1/2	61.1079	95.8672	154.149	115.137	148.724	212.333
	(61.108)	(95.867)	(154.15)	(115.14)	(148.72)	(212.33)[a]
	1 1	1 2	1 3	1 1	1 2	1 3
1	24.2982	59.0860	60.0631	38.9820	77.6243	92.3673
	(24.298)	(59.086)	(60.063)	(38.982)	(77.624)	(92.367)[a]
	1 1	1 2	2 1	1 1	1 2	2 1
2	14.8424	23.9449	38.7839	19.6050	33.4593	54.2224[a]
	(14.842)	(23.945)	(38.784)	(19.605)	(33.459)	(54.222)
	1 1	2 1	3 1	1 1	2 1	3 1

[a] Ω^*_{mn} values from Ref. 30.

Table 4 Semianalytical DQ solution results for the first six mode frequencies of specially orthotropic plates

λ	α	Dimensionless frequency Ω^*_{mn} and mode numbers m n					
		L/T orientation					
1/2	0.0	18.0446	20.9150	26.8885	36.4868	50.9416	49.7123[a]
	0.4	19.6117	23.3307	30.8588	42.6517	57.5286	58.6170
		1 1	1 2	1 3	1 4	2 1	1 5
1	0.0	20.9150	36.4868	54.0543	66.4239	68.3763	95.8476
	0.4	23.3307	42.6517	61.4341	78.5460	79.2194	112.893
		1 1	1 2	2 1	1 3	2 2	2 3
2	0.0	36.4868	68.3763	109.875	119.675	137.279	184.790
	0.4	42.6517	79.2194	129.630	139.179	163.303	218.588
		1 1	2 1	1 2	3 1	2 2	3 2
		T/L orientation					
1/2	0.0	13.7647	18.9426	29.7069	36.5075	41.5126	46.0876
	0.4	15.4221	21.8590	34.9369	41.4275	47.6227	54.4983
		1 1	1 2	1 3	2 1	2 2	1 4
1	0.0	18.9426	41.5126	46.0876	66.1001	76.1026	94.5977
	0.4	21.8590	47.6227	54.4983	77.7446	87.7491	111.268
		1 1	2 1	1 2	2 2	3 1	1 3
2	0.0	46.0876	66.1001	99.3211	145.296	163.321	179.985
	0.4	54.4983	77.7446	116.180	169.906	189.790	215.799
		1 1	2 1	3 1	4 1	1 2	2 2

[a] Note that in this case the last two frequencies are of the sixth and fifth modes, respectively.

respect to the increase in the number of sampling points. The calculated results were compared with the available results of uniform thickness isotropic plates,[1] linearly tapering thickness isotropic plates,[28] and uniform thickness orthotropic plates.[29] In all cases, the agreement between the compared results was found to be excellent. However, the convergence and accuracy of the proposed solution method were investigated for the boundary conditions of the present problem as well; sample results of the same are given in Tables 2 and 3. In Table 2, frequency values are given to eight significant digits to illustrate the convergence pattern; it may be seen that the solution method exhibits monotonic convergence with increasing number of sampling points. Based on such convergence studies, further quadrature solution results, such as given in Tables 3 and 4, were obtained with the number of sampling points N = 15.

To establish the accuracy of the proposed solution method, the results of available literature[30] were utilized. These results are based on a highly accurate semianalytical approximate method utilizing power series expansions and provide the frequencies of first three modes of free vibrations of uniform thickness and tapered rectangular plates having boundary conditions identical to those of the present work. Table 3 gives a comparison of the results from the present solution method with some of the data of the cited work.[30] In this work, the dimensionless rotational stiffness and frequency (denoted by symbols with asterisks) were defined as $K_\theta^* = k_\theta b/D_0$ and $\Omega^* = \omega\sqrt{(\rho h_0 b^4/D_0)}$, respectively. Also, the frequency values were given to five significant digits. It may be noted that in order to keep the comparison meaningful, the dimensionless rotational stiffness and frequency in Table 3 are in terms of K_θ^* and Ω^*, respectively, and the frequency values of the present calculations are given to six significant digits. It is apparent that the calculated results are in excellent agreement with the published data[30]; this, indeed, confirms the high numerical accuracy of the proposed solution method.

The present investigators are not familiar with any published work on specially orthotropic plates with the boundary conditions of the present work. Table 3 includes some of such results. Some more results, however, though still limited, are given in Table 4. These results include frequencies of first six free vibration modes of uniform and linearly tapered thickness plates. It is believed that these results are new and should be useful for future research work in this area.

The computer programs for the present work were developed and executed on DEC-stations 5000/25 (operating system: Ultrix 4.2a) at the University of Oklahoma. The average CPU time for obtaining 10 different mode frequencies of a given plate has been found to be less than 0.10 s which shows the capability of fast computations by the present solution method.

Closure

The present investigators believe that there is an incontestable need for development of solution methods for determination of the characteristics of basic machine and structural elements that could be used for real-time design and analysis purposes. In this context, plates are such elements, which due to boundless parametric variations in real practical applications offer considerable challenge. In this paper, a differential quadrature solution with semianalytical approach was proposed for the free vibration analysis of rectangular isotropic and orthotropic tapered plates. The semianalytical approach utilized herein has the identical basis as that employed in other semianalytical numerical methods, such as the semianalytical finite element method, i.e., the widely known finite strip method.[31] To the best of the knowledge of the present investigators, however, differential quadrature solutions utilizing the semianalytical approach have not been published earlier.

The results contained in this paper indicate that the proposed solution method has considerable promise for its development for real-time analysis and design purposes. This is because the calculated results are numerically accurate and are obtained in very small CPU times.

In this work, plates having two opposite edges simply supported were considered. A semianalytical approach may generally be implemented if the field variable can be expressed, complying with the boundary conditions, analytically in one or more spatial directions. Consequently, utilizing such semianalytical expressions, one can reduce the number of spatial variables in the differential equation; the commonly used technique for this purpose is the method of Kantorovich and Krylov.[32] Thus, semianalytical differential quadrature solutions for the boundary conditions, other than simply supported, on two opposite edges are seemingly possible; work in this direction is being carried out by the present investigators.

The results reported in this paper were for linearly tapered plates. The analysis, however, was general with respect to the thickness function $H(X)$ in that the plate can have any form of thickness variation in the x direction. Of course, plates with discontinuous thickness distribution (for example, stepped plates) may also be analyzed; in such cases, however, a sampling point should be located at each point of discontinuity. This situation may be handled more conveniently by domain decomposition. The vibration analysis of stepped plates by the usual differential quadrature method in conjunction with domain decomposition is available elsewhere.[19] In fact, implementing the domain decomposition in the semianalytical DQ solution would be much simpler than that in the usual differential quadrature solution.

References

[1]Leissa, A. W., "The Free Vibration of Rectangular Plates," *Journal of Sound and Vibration*, Vol. 31, No. 3, 1973, pp. 257–293.

[2]Bellman, R., and Casti, J., "Differential Quadrature and Long-Term Integration," *Journal of Mathematical Analysis and Applications*, Vol. 34, No. 2, 1971, pp. 235–238.

[3]Bellman, R., Kashef, B. G., and Casti, J., "Differential Quadrature: A Technique for the Rapid Solution of Nonlinear Partial Differential Equations," *Journal of Computational Physics*, Vol. 10, No. 1, 1972, pp. 40–52.

[4]Mingle, J. O., "Computational Considerations in Non-Linear Diffusion," *International Journal for Numerical Methods in Engineering*, Vol. 7, No. 1, 1973, pp. 103–116.

[5]Civan, F., and Sliepcevich, C. M., "Application of Differential Quadrature to Transport Processes," *Journal of Mathematical Analysis and Applications*, Vol. 93, No. 1, 1983, pp. 206–221.

[6]Civan, F., and Sliepcevich, C. M., "Solving Integro-Differential Equations by the Quadrature Method," *Integral Methods in Science and Engineering*, Hemisphere, New York, 1986, pp. 106–113.

[7]Bert, C. W., Jang, S. K., and Striz, A. G., "Two New Approximate Methods for Analyzing Free Vibrations of Structural Components," *AIAA Journal*, Vol. 26, No. 5, 1988, pp. 612–618.

[8]Jang, S. K., Bert, C. W., and Striz, A. G., "Application of Differential Quadrature to Static Analysis of Structural Components," *International Journal for Numerical Methods in Engineering*, Vol. 28, No. 3, 1989, pp. 561–577.

[9]Bert, C. W., Wang, X., and Striz, A. G., "Differential Quadrature for Static and Free Vibration Analysis of Anisotropic Plates," *International Journal of Solids and Structures*, Vol. 30, No. 13, 1993, pp. 1737–1744.

[10]Shu, C., and Richards, B. E., "Application of Generalized Differential Quadrature to Solve Two-Dimensional Incompressible Navier–Stokes Equations," *International Journal for Numerical Methods in Fluids*, Vol. 15, No. 7, 1992, pp. 791–798.

[11]Striz, A. G., and Chen, W., "Application of the Differential Quadrature Method to the Driven Cavity Problem," *International Journal of Non-Linear Mechanics*, Vol. 29, No. 5, 1994, pp. 665–670.

[12]Shu, C., Khoo, B. C., Yeo, K. S., and Chew, Y. T., "Application of GDQ Scheme to Simulate Natural Convection in a Square Cavity," *International Communications in Heat and Mass Transfer*, Vol. 21, No. 6, 1994, pp. 809–817.

[13]Malik, M., and Bert, C. W., "Differential Quadrature Solution for Steady State Incompressible and Compressible Lubrication Problems," *Journal of Tribology*, Vol. 116, No. 2, 1994, pp. 296–302.

[14]Malik, M., and Civan, F., "A Comparative Study of Differential Quadrature and Cubature Methods Vis-a-Vis Some Conventional Techniques in Context of Convection-Diffusion-Reaction Problems," *Chemical Engineering Science*, Vol. 50, No. 3, pp. 531–547.

[15]Bellman, R., *Methods of Nonlinear Analysis*, Vol. 2, Academic, New York, 1973.

[16]Bellman, R., and Adomian, G., *Partial Differential Equations*, D. Reidel, Dordrecht, The Netherlands, 1985.

[17]Bellman, R., and Roth, R. S., *Methods in Approximation*, D. Reidel, Dordrecht, The Netherlands, 1986.

[18]Zwillinger, D., *Handbook of Differential Equations*, Academic, San Diego, CA, 1992.

[19]Chen, W. L., "A New Approach for Structural Analysis: The Quadrature Element Method," Ph.D. Dissertation, Univ. of Oklahoma, Norman, OK, 1994.

[20]Striz, A. G., Chen, W., and Bert, C. W., "Static Analysis of Structures by the Quadrature Element Method," *International Journal of Solids and Structures*, Vol. 31, 1994, pp. 2807–2818.

[21]Bathe, K.-J., *Finite Element Procedures in Engineering Analysis*, Prentice–Hall, Englewood Cliffs, NJ, 1982.

[22]Quan, J. R., and Chang, C. T., "New Insights in Solving Distributed System Equations by the Quadrature Method—I. Analysis," *Computers in Chemical Engineering*, Vol. 13, No. 7, 1989, pp. 779–788.

[23]Quan, J. R., and Chang, C. T., "New Insights in Solving Distributed System Equations by the Quadrature Method—II. Numerical Experiments," *Computers in Chemical Engineering*, Vol. 13, No. 9, 1989, pp. 1017–1024.

[24]Bert, C. W., Wang, X., and Striz, A. G., "Convergence of the DQ Method in the Analysis of Anisotropic Plates," *Journal of Sound and Vibration*, Vol. 170, No. 1, 1994, pp. 140–144.

[25]Malik, M., "Differential Quadrature Method in Computational Mechanics: New Developments and Applications," Ph.D. Dissertation, Univ. of Oklahoma, Norman, OK, 1994.

[26]Bert, C. W., and Malik, M., "Free Vibration Analysis of Thin Cylindrical Shells by the Differential Quadrature Method," *Journal of Pressure Vessel Technology* (to be published).

[27]Vinson, J. R., and Sierakowski, R. L., *The Behavior of Structures Composed of Composite Materials*, Martinus Nijhoff, Dordrecht, The Netherlands, 1986.

[28]Bhat, R. B., Laura, P. A. A., Gutierrez, R. G., Cortinez, V. H., and Sanzi, V. H., "Numerical Experiments on the Determination of Natural Frequencies of Transverse Vibrations of Rectangular Plates of Nonuniform Thickness," *Journal of Sound and Vibration*, Vol. 138, No. 2, 1990, pp. 205–219.

[29]Huffington, N. J., Jr., and Hoppmann, W. H., II, "Closure to Discussion on 'On the Transverse Vibrations of Rectangular Orthotropic Plates,'" *Journal of Applied Mechanics*, Vol. 26, No. 2, 1959, pp. 308, 309.

[30]Kobayashi, H., and Sonoda, K., "Vibration and Buckling of Tapered Rectangular Plates with Two Opposite Edges Simply Supported and the Other Two Edges Elastically Restrained Against Rotation," *Journal of Sound and Vibration*, Vol. 146, No. 2, pp. 323–337.

[31]Cheung, Y. K., *Finite Strip Method in Structural Analysis*, Pergamon, New York, 1976.

[32]Kantorovich, L. V., and Krylov, V. I., *Approximate Methods of Higher Analysis*, Interscience, New York, 1958.